

## MINIMAL $K$ -TYPES FOR $G_2$ OVER A $p$ -ADIC FIELD

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**ABSTRACT.** We single out certain representations of compact open subgroups of  $G_2$  over a  $p$ -adic field and show they play a role in the representation theory of  $G_2$  similar to minimal  $K$ -types in the theory of real groups.

**Introduction.** Let  $F$  be a  $p$ -adic field. The purpose of this paper is to define nondegenerate representations for the group  $G = G_2$ . The importance of these representations for the representation theory of  $G$  is analogous to the role played by Vogan's theory of minimal  $K$ -types for real groups. In particular, it will be shown that every irreducible admissible representation  $(\pi, V)$  of  $G$  contains a nondegenerate representation and moreover, nondegenerate representations provide a means of partitioning the irreducible representations of  $G$ .

A nondegenerate representation is a pair  $(L, \Omega)$  consisting of an open compact subgroup  $L$  of  $G$ , and a representation  $\Omega$  of  $L$  satisfying a certain cuspidality or semisimplicity property. As was the case for  $U(2, 1)$  and  $GS p(4)$  in [M1, M2], a general theory for describing these representations is missing and so instead we merely list the nondegenerate representations. This is done in §1. In §2, it is shown by methods similar to those of [M1, M2] that any irreducible admissible representation of  $G$  contains a nondegenerate representation. The irreducible admissible representations of  $G$  which contain a given  $(L, \Omega)$  can be determined by an investigation of the representation theory of the Hecke algebra  $\mathcal{H}(G//L, \Omega^t)$ . It is likely the description of  $\mathcal{H}(G//L, \Omega^t)$  will follow along the lines given in [HM1, M1, M2].

**1. Nondegenerate representations.** Let  $F$  be a  $p$ -adic field and let  $R$  be its ring of integers. Let  $\mathfrak{p}$  be the prime ideal of  $R$  and  $\varpi$  a prime element in  $\mathfrak{p}$ . Let  $G$  be the algebraic group  $G_2$  and let  $G(F)$  be the group of  $F$ -rational points of  $G$ . The group  $G$  has a well-known seven-dimensional representation which we now review. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . It will be convenient for us to assume that the residue characteristic  $p \neq 2, 3$ . The Lie algebra  $\mathfrak{g}$  has the seven-dimensional representation. See (1.1). The author is grateful to G. Seligman for showing him this explicit representation of  $\mathfrak{g}$ ; see [S]. In this realization, the set  $\mathfrak{a}$  of diagonal matrices in  $\mathfrak{g}$  is a Cartan subalgebra corresponding to a maximal split torus  $A$  of  $G$ .

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$$(1.1) \quad \mathfrak{g} = \left\{ \begin{bmatrix} A & X & Y & -Z & N & -J & 0 \\ x & B & -M & Y & Z & 0 & J \\ y & -m & A-B & -X & 0 & -Z & -N \\ -2z & 2y & -2x & 0 & 2X & -2Y & 2Z \\ n & z & 0 & x & B-A & M & -Y \\ -j & 0 & -z & -y & m & -B & -X \\ 0 & j & -n & z & -y & -x & -A \end{bmatrix} \right\}.$$

The set of upper triangular matrices  $\mathfrak{b}$  corresponds to a Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$ . Denote by  $E_{i,j}$  the  $7 \times 7$  matrix whose  $r, s$  entry is  $\delta_{r,i}\delta_{s,j}$  (Kronecker delta). Abbreviate  $E_{i,i}$  to  $E_i$ . The Cartan subalgebra  $\mathfrak{a}$  is of course spanned by the two vectors

$$E_A = E_1 + E_3 - E_5 + E_7, \quad E_B = E_2 - E_3 + E_5 - E_6.$$

The group  $A = \mathbf{A}(F)$  is equal to the set of  $7 \times 7$  matrices of the form

$$(1.2) \quad d(s, t) = sE_1 + tE_2 + st^{-1}E_3 + E_4 + s^{-1}tE_5 + t^{-1}E_6 + s^{-1}E_7,$$

$s, t \in F^\times$ . The group  $\mathbf{A}(R)$  consists of those  $d(s, t)$  with  $s, t \in R^\times$ . For  $i$  a positive integer, set

$$A_i = \{d(s, t) \mid s, t = 1 \pmod{\mathfrak{p}^i}\}.$$

Define linear functionals  $a$  and  $b$  on  $\mathfrak{a}$  by

$$a(rE_A + sE_B) = r - s, \quad b(rE_A + sE_B) = -r + 2s,$$

and set

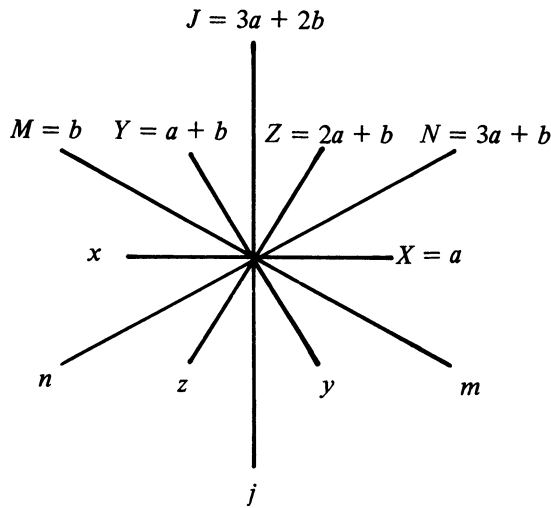
$$\begin{aligned} E_X = E_a &= E_{1,2} - E_{3,4} + 2E_{4,5} - E_{6,7}, \\ E_M = E_b &= -E_{2,3} + E_{5,6}, \\ E_Y = E_{a+b} &= E_{1,3} + E_{2,4} - 2E_{4,6} - E_{5,7}, \\ E_Z = E_{2a+b} &= -E_{1,4} + E_{2,5} - E_{3,6} + 2E_{4,7}, \\ E_N = E_{3a+b} &= E_{1,5} - E_{3,7}, \\ E_J = E_{3a+2b} &= -E_{1,6} + E_{2,7}, \\ E_x = E_{-a} &= E_{2,1} - 2E_{4,3} + E_{5,4} - E_{7,6}, \\ E_m = E_{-b} &= -E_{3,2} + E_{6,5}, \\ E_y = E_{-a-b} &= E_{3,1} + 2E_{4,2} - E_{6,4} - E_{7,5}, \\ E_z = E_{-2a-b} &= -2E_{4,1} + E_{5,2} - E_{6,3} + E_{7,4}, \\ E_n = E_{-3a-b} &= E_{5,1} - E_{7,3}, \\ E_j = E_{-3a-2b} &= -E_{6,1} + E_{7,2}. \end{aligned}$$

Each of the above twelve vectors is a root vector, i.e.  $[h, E_c] = c(h)E_c$ . Denote the one-dimensional root space of  $\mathfrak{g}$  spanned by  $E_c$  by  $\mathfrak{u}_c$ . The twelve linear functionals

$$\Phi = \{\pm a, \pm b, \pm(a+b), \pm(2a+b), \pm(3a+b), \pm(3a+2b)\}$$

form a root system of type  $G_2$ .

Given roots  $f$  and  $d$ , there is an integer  $C_{f,d}$  such that  $[E_f, E_d] = C_{f,d}E_{f+d}$ . Indeed, we have  $[E_a, E_b] = -E_{a+b}$ ,  $[E_a, E_{a+b}] = -2E_{2a+b}$ ,  $[E_a, E_{2a+b}] = 3E_{3a+b}$ ,  $[E_b, E_{3a+b}] = E_{3a+2b}$ ,  $[E_{a+b}, E_{2a+b}] = 3E_{3a+2b}$ . The root subspaces  $\mathfrak{u}_c$  can be



exponentiated to yield root groups  $U_c$  inside  $G$ . Let  $u_c: F \rightarrow U_c(F)$  be the exponential map. For example, if  $I$  is the identity matrix, then  $u_X(t) = I + tE_X - t^2E_{3,5}$ . The filtration of  $F$  by powers of the prime ideal transfers under  $u_c$  to a filtration of the root group  $U_c = U_c(F)$ . Set

$$u_{c,i} = \varpi^i E_c \quad \text{and} \quad U_{c,i} = u_c(\varpi^i).$$

The Borel subgroup  $B$  is equal to the product of the groups  $A$  and  $U_c$ ,  $c$  a positive root. Let  $N$  be the normalizer of  $A$ . For  $c$  a root, set

$$w_c(t) = u_c(t)u_{-c}(-t^{-1})u_c(t).$$

The  $w_c(t)$ 's and  $A$  generate the group  $N = N(F)$ . The Weyl group  $W = N/A$  is generated by the reflections

$$s_X = w_X(1) \quad \text{and} \quad s_M = w_M(1).$$

The affine Weyl group  $W^{\text{aff}} = N/A(R)$  is generated by  $s_X$ ,  $s_M$  and the reflection

$$s_j = w_j(\varpi^{-1}).$$

We briefly review some basic facts about parahoric subgroups of  $G$ . The group generated by  $A(R)$ ,  $U_{c,0}$  ( $c > 0$ ), and  $U_{c,1}$  ( $c < 0$ ), which we write as

$$B = A(R) \prod_{c>0} U_{c,0} \prod_{c<0} U_{c,1},$$

is an Iwahori subgroup of  $G$ . It consists of those elements of  $K = G(R)$  which are upper triangular modulo  $\varpi$ . The parahoric subgroups of  $G$  are those subgroups of  $G$  which contain  $B$ . It is known that a parahoric subgroup  $P$  is generated by  $B$  and a subset  $I$  of  $S = \{s_X, s_M, s_j\}$ .

The parahoric subgroup  $P_I$  generated by  $B$  and  $I$  is compact if and only if  $I$  is a proper subset of  $S$ . In fact, up to conjugacy the maximal compact subgroups of  $G$  are the parahoric subgroups associated to a maximal proper subset of  $S$ . So, the three conjugacy classes of maximal compact subgroups are

$$K = P_{\{s_X, s_M\}}, \quad K' = P_{\{s_X, s_j\}}, \quad \text{and} \quad K'' = P_{\{s_M, s_j\}}.$$

If  $P$  is a compact parahoric subgroup, let  $P_1$  denote the maximal normal pro- $p$ -subgroup of  $P$ . We call  $P_1$  the radical of  $P$ . For example,  $K_1$  consists of those elements in  $K$  which are congruent to 1 mod  $\varpi$ .

The group  $P/P_1$  is easily described in terms of the Dynkin diagram

$$(1.3) \quad \bigcirc_{s_j} - \bigcirc_{s_M} \equiv \bigcirc_{s_X}$$

of  $S$ . Let  $I$  be the subset of  $S$  associated to  $P$ , and let  $\mathbb{F}_q$  be the finite field  $R/\varpi$  with  $q$  elements. Then,  $P/P_1$  is the rational points of a reductive group over  $\mathbb{F}_q$  whose Dynkin diagram is the subdiagram of (1.3) associated to  $I$ . With these preliminaries aside, we define a **nondegenerate representation of level one** to be a compact parahoric subgroup  $P$  and a representation  $\Omega$  of  $P$  trivial on  $P_1$  so that  $\Omega$  is cuspidal as a representation of  $P/P_1$ .

To each parahoric subgroup  $P$ , it will be useful to associate a filtration of  $P$  due to Prasad and Raghunathan [PR]. In preparation for this, we review the notions of affine roots and heights. Let  $\mathbf{X}$  be the  $\mathbb{Z}$ -span of the simple roots  $a$  and  $b$ . An affine root is an element  $\beta = (c, i)$  in the additive group  $\mathbf{X} \times \mathbb{Z}$  subject to the condition that  $c \in \Phi \cup \{0\}$ . If  $c \in \Phi$ , define the affine root group  $U_\beta$  to be  $U_\beta = U_{c,i}$ . If  $c = 0$ , and  $i \geq 1$ , define

$$U_{0,i} = \{d(s, t) \mid s, t = 1 \bmod \varpi^i\}.$$

We also define analogous notions in  $\mathfrak{g}$ , i.e.,  $u_\beta = u_{c,i} = \varpi^i E_c$ ,  $c \in \Phi$ , and

$$u_{0,i} = \varpi^i E_A + \varpi^i E_B, \quad i \geq 0.$$

Each affine root  $\beta$  has a unique decomposition in terms of the simple affine roots  $(b, 0)$ ,  $(a, 0)$ , and  $(-3a - 2b, 0)$ ,

$$\beta = h_0(\beta)(b, 0) + h_1(\beta)(a, 0) + h_2(\beta)(-3a - 2b, 0).$$

As an example,  $(0, 1) = 2(b, 0) + 3(a, 0) + 1(-3a - 2b, 0)$ . For  $I \subseteq S$ , define a height function  $\text{ht}_I$  on the affine roots by the sum

$$\text{ht}_I(\beta) = \sum' h_i(\beta)$$

over those  $i$  for which  $s_i \notin I$ . Then, the radical of  $P_I$  is equal to

$$P_{I,1} = \text{subgroup of } G \text{ generated by } U_\beta \text{ such that } \text{ht}_I(\beta) \geq 1.$$

More generally, if  $t$  is a positive integer, set

$$(1.4) \quad P_{I,t} = \text{subgroup of } G \text{ generated by } U_\beta \text{ such that } \text{ht}_I(\beta) \geq t.$$

The  $P_{I,t}$ 's are a filtration of normal subgroups in  $P_I$ . We define similar lattices, i.e.  $R$ -submodules, in  $\mathfrak{g}$  by

$$\mathcal{P}_{I,t} = R\text{-submodule of } \mathfrak{g} \text{ generated by } u_\beta \text{ such that } \text{ht}_I(\beta) \geq t.$$

The  $\mathcal{P}_{I,t}$ 's satisfy the periodicity relation

$$\mathcal{P}_{I,t+\text{ht}_I(0,1)} = \varpi \mathcal{P}_{I,t}.$$

We extend the definition of  $\mathcal{P}_{I,t}$  to all  $t \in \mathbb{Z}$  via periodicity. The Cayley transform  $\mathcal{C}: \mathfrak{g} \rightarrow G$  given by

$$\mathcal{C}(x) = (1 - x)(1 + x)^{-1}$$

maps  $\mathcal{P}_{I,t}$  bijectively to  $P_{I,t}$  when  $t > 0$ , and induces, for  $t \geq t' > 0$ , an isomorphism

$$(1.5) \quad \mathcal{P}_{I,t}/\mathcal{P}_{I,t+t'} \rightarrow P_{I,t}/P_{I,t+t'}.$$

When  $I = \{s_X, s_M\}$ , i.e.  $P_I = K = \mathbf{G}(R)$ , the filtration subgroup  $K_i$  is just the  $i$ th principal congruence subgroup of  $K$ .

Given a lattice  $\mathcal{M} \subseteq \mathfrak{g}$ , let

$$\mathcal{M}^* = \{x \in \mathfrak{g} \mid \mathrm{tr}(x\mathcal{M}) \in \mathcal{O}\}.$$

In particular, we have  $\mathcal{P}_{I,t}^* = \mathcal{P}_{I,-t+1}$ .

The isomorphism (1.5) allows us to identify the characters of  $P_{I,t}/P_{I,t+t'}$  with  $\mathcal{P}_{I,-t-t'+1}/\mathcal{P}_{I,-t+1}$ . To realize this identification, consider the seven-dimensional representation of  $\mathfrak{g}$  given in (1.1). The form

$$\langle x, y \rangle = \mathrm{tr}(xy), \quad x, y \in \mathfrak{g},$$

is a multiple of the Killing form on  $\mathfrak{g}$ . If  $\psi$  is an additive character of  $F$  with conductor  $\mathcal{O}$ , and  $\alpha \in \mathcal{P}_{I,-t-t'+1}/\mathcal{P}_{I,-t+1}$ , define  $\Omega_\alpha$  to be the character

$$(1.6) \quad \Omega_\alpha(\mathcal{E}(x)) = \psi(\langle x, -\alpha \rangle / 2), \quad x \in \mathcal{P}_{I,t}, \alpha \in \alpha.$$

We are ready to define the nondegenerate representations of unramified type. Fix a positive integer  $i$  and consider  $K = P_{\{s_X, s_M\}}$ . Abbreviate  $\mathcal{P}_{\{s_X, s_M\},t}$  to  $\mathcal{K}_t$ . By (1.6),  $\{K_j/K_{j+1}\}^\wedge = \mathcal{K}_{-i}/\mathcal{K}_{-i+1} = \mathfrak{g}(\mathbf{F}_q)$ . Given such a character coset  $\alpha$ , let  $\alpha = \mathbf{s} + \mathbf{n}$  be the Jordan decomposition of  $\alpha$  into its semisimple and nilpotent parts (in  $\mathfrak{g}(\mathbf{F}_q)$ ). We shall momentarily define a group  $L$  and a representation  $\Omega_\mathbf{s}$  of  $L$ . The collection of  $(L, \Omega_\mathbf{s})$ 's and their  $G$ -conjugates will be the nondegenerate representations of unramified type. To define the group  $L$ , which turns out to be a parahoric filtration subgroup, we need to recall certain Levi subalgebras of  $\mathfrak{g}$ .

Given a subset  $I$  of  $\{s_X, s_M\}$ , let  $\mathfrak{p}_I$  be the parabolic subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{b}$  and  $\mathrm{Ad}(I) \cdot \mathfrak{b}$ . The subalgebra  $\mathfrak{p}_I$  can be written as  $\mathfrak{m}_I \mathfrak{u}_I$ , where  $\mathfrak{u}_I$  is the nilpotent radical of  $\mathfrak{p}_I$ , and  $\mathfrak{m}_I$  is a Levi component of  $\mathfrak{p}_I$ , which is stable under the Cartan involution  $\theta$  of  $\mathfrak{g}$  given by

$$(1.7) \quad \theta(E_c) = -E_{-c}, \quad c \text{ a root}; \quad \theta(E_A) = -E_A, \quad \theta(E_B) = -E_B.$$

For example,  $\mathfrak{m}_{\{s_X, s_M\}} = \mathfrak{g}$  and  $\mathfrak{m}_\emptyset = \mathfrak{a}$ . By replacing  $\alpha$  by a conjugate under  $K$ , we may assume  $\mathbf{n}$  is a sum of positive root vectors and  $\mathbf{s} \in \mathfrak{m}_I$  for some  $I \subseteq \{s_X, s_M\}$ , and  $\mathbf{s} \notin \mathfrak{m}_{I'}$  whenever  $I'$  is a proper subset of  $I$ . Let  $\underline{\mathfrak{u}}_I = \theta(\mathfrak{u}_I)$  be the opposite nilpotent subalgebra of  $\mathfrak{u}_I$ . Denote by  $\mathbf{M}$ ,  $\mathbf{U}$ , and  $\underline{\mathbf{U}}$  the subgroups of  $\mathbf{G}$  whose Lie algebras are  $\mathfrak{m}_I$ ,  $\mathfrak{u}_I$ , and  $\underline{\mathfrak{u}}_I$  respectively. Denote the  $F$ -rational points of the corresponding groups by  $M$ ,  $U$ , and  $\underline{U}$  and let  $M_i$ ,  $U_i$ , and  $\underline{U}_i$  be the intersections of these latter groups with  $K_i$ . Define  $L \subseteq K_i$  by

$$L = \underline{U}_{i+1} M_i U_i = P_{I, \mathrm{ht}_I(0,1)i}.$$

The restriction of the character  $\Omega_\alpha$  to  $L$  does not depend on  $\mathbf{n}$ , i.e. if  $\alpha' = \mathbf{s}' + \mathbf{n}'$  with  $\mathbf{s} = \mathbf{s}'$ , then  $\Omega_\alpha = \Omega_{\alpha'}$  on  $L$ . Denote this character by  $\Omega_\mathbf{s}$ . For  $\mathbf{s}$  nonzero, define the representations  $(L, \Omega_\mathbf{s})$  and their  $G$ -conjugates to be the **nondegenerate representations of unramified type**.

In order to describe the nondegenerate representations of ramified type, we recall the conjugacy classes of nonzero nilpotent elements in  $\mathfrak{g}(\mathbb{F}_q)$ . There are six such classes [C]. They are

$$(1.8) \quad \begin{aligned} (a) \quad \mathbf{n}_a &= E_X + E_M \bmod \mathcal{P}, \\ (b) \quad \mathbf{n}_b &= E_M + E_Z \bmod \mathcal{P}, \\ (c) \quad \mathbf{n}_c &= E_M + \varepsilon E_Z \bmod \mathcal{P}, \quad \varepsilon \text{ a nonsquare in } \mathbb{F}_q, \\ (d) \quad \mathbf{n}_d &= \begin{cases} E_M + \mu E_N \bmod \mathcal{P}, & \text{when } q \equiv 1 \pmod{3} \text{ and } \\ & \mu \text{ is a noncube,} \\ E_M - E_Z + \zeta E_N \bmod \mathcal{P}, & \text{when } q \equiv 2 \pmod{3} \text{ and } \\ & x^3 - 3x - \zeta \text{ is irreducible,} \end{cases} \\ (e) \quad \mathbf{n}_e &= E_Z \bmod \mathcal{P}, \\ (f) \quad \mathbf{n}_f &= E_J \bmod \mathcal{P}. \end{aligned}$$

In order to simplify notation for  $t \in \mathbb{Z}$ , let

$$\ell^t = \mathcal{P}_{\emptyset, t}, \quad \not\mathcal{P}^t = \mathcal{P}_{\{s_X\}, t}, \quad \text{and} \quad \varphi^t = \mathcal{P}_{\{s_J\}, t}.$$

To each nilpotent element  $\mathbf{n}$  in (1.8), and positive integer  $i$ , we associate a finite number of triples  $(m, m_+, \alpha)$  consisting of two lattices  $m \subset m_+$  in  $\mathfrak{g}$  and a coset  $\alpha$  of  $m_+/m$ , such that

$$(1.9) \quad \begin{aligned} (i) \quad & \varpi^i \alpha \bmod \mathcal{P} \text{ is a nonzero multiple of } \mathbf{n}, \text{ and} \\ (ii) \quad & \text{the coset } \alpha \text{ contains no nilpotent elements.} \end{aligned}$$

For each triple, we construct a compact open subgroup  $L \subset G$  and a representation  $\Omega$  of  $L$ . The pair  $(L, \Omega)$  will be a nondegenerate representation of ramified type.

$$(1.10a) \quad \begin{aligned} m_+ &= \ell^{-6i+1}, \quad m = \ell^{-6i+2}, \\ \alpha &= \varpi^{-i} \{u\varpi E_j + \mathbf{n}_a\} \bmod m, \quad u \in R^\times. \end{aligned}$$

$$(1.10b,c,d.1) \quad \begin{aligned} m_+ &= \not\mathcal{P}^{-3i+1}, \quad m = \not\mathcal{P}^{-3i+2}, \\ \alpha &= \varpi^{-i} \{u\varpi E_j + \mathbf{n}\} \bmod m, \quad u \in R^\times, \mathbf{n} \in \{\mathbf{n}_b, \mathbf{n}_c, \mathbf{n}_d\}. \end{aligned}$$

That is, case (1.10b.1) is when  $\mathbf{n}$  is  $\mathbf{n}_b$ , and so forth.

$$(1.10b,c,d.2) \quad \begin{aligned} m_+ &= \varphi^{-2i+1}, \quad m = \varphi^{-2i+2}, \\ \alpha &= \varpi^{-i} \{\varpi(uE_z + vE_y + wE_m) + \mathbf{n}\} \bmod m, \quad u \in R^\times, \mathbf{n} \in \{\mathbf{n}_b, \mathbf{n}_c, \mathbf{n}_d\}. \end{aligned}$$

$$(1.10e) \quad \begin{aligned} m_+ &= \varphi^{-2i+1}, \quad m = \varphi^{-2i+2}, \\ \alpha &= \varpi^{-i} \{\varpi(tE_n + uE_z + wE_m) + \mathbf{n}_e\} \bmod m, \\ &\quad \begin{bmatrix} u & 0 & -w \\ t & -u & 0 \end{bmatrix} \text{ has rank 2 mod } \mathcal{P}. \end{aligned}$$

To continue, we introduce some notation. Define a sequence of lattices in  $\mathfrak{g}$  by

$$\begin{aligned} c_4 &= \varpi \not\mathcal{P}^0, \\ c_3 &= c_4 + u_{n,1} + u_{z,1} + u_{y,1} + u_{m,1}, \\ c_2 &= c_3 + u_{j,1} + u_{J,0}, \\ c_1 &= c_2 + u_{N,0} + u_{Z,0} + u_{Y,0} + u_{M,0}, \\ c_k &= \varpi c_{k-4}. \end{aligned}$$

$$\begin{aligned}
 (1.10f.1) \quad & m_+ = c_{-4i+2}, \quad m = c_{-4i+3}, \\
 & \alpha = \varpi^{-1}\{\mathbf{n}_f + \varpi u E_j\}, \quad u \in R^\times. \\
 (1.10f.b,c,d) \quad & m_+ = \not\varphi^{-3i+2}, \quad m = \not\varphi^{-3i+3}, \\
 & \alpha = \varpi^{-i}\{u\mathbf{n}_f + \varpi\theta(\mathbf{n})\}, \quad u \in R^\times, \mathbf{n} \in \{\mathbf{n}_b, \mathbf{n}_c, \mathbf{n}_d\}. \\
 (1.10f.2) \quad & m_+ = \ell^{-6i+5}, \quad m = \ell^{-6i+6}, \\
 & \alpha = \varpi^{-i}\{\varpi(uE_j + vE_m) + \mathbf{n}_f\} \bmod m, \quad u, v \in R^\times. \\
 (1.10f.3) \quad & m_+ = \mathcal{P}_{\{s_M, s_j\}, -3i+2} \quad (i \geq 2), \quad m = \mathcal{P}_{\{s_M, s_j\}, -3i+3}, \\
 & \alpha = \varpi^{-i}\{aE_j + \varpi(cE_A + dE_B + eE_M + fE_n) + \varpi^2(gE_n + hE_j)\} \\
 & \quad \bmod \mathcal{P}_{\{s_M, s_j\}, -3i+2}, \\
 & a \in R^\times \text{ and the characteristic polynomial of} \\
 & \quad \begin{bmatrix} c & 0 & -a \\ -g & c-d & e \\ h & f & -d \end{bmatrix} \bmod \not\varphi \text{ is irreducible.} \\
 (1.10f.4) \quad & m_+ = \mathcal{P}_{\{s_X, s_j\}, -2i+2} \quad (i \geq 2), \quad m = \mathcal{P}_{\{s_X, s_j\}, -2i+3}, \\
 & \alpha = \varpi^{-i}\{n_g + \varpi(aE_A + bE_B + cE_X + dE_x) + \varpi^2vE_j\} \bmod m, \\
 & \quad (a+b)^2 - 4v \text{ a nonsquare mod } \not\varphi.
 \end{aligned}$$

It is clear that (1.9i) holds for each of the cases above. Consider property (1.9ii). In case (1.10a), for any  $\alpha \in \alpha$ , we have  $\text{trace}\{\varpi^{-1}(\varpi^i\alpha)^6\} = 24u \bmod \not\varphi$ . Thus, property (1.9ii) is obvious. The other cases are similar.

For each triple in (1.10), let

$$\mathcal{L} = m^*, \quad \mathcal{L}_+ = m_+^* \quad \text{and} \quad L = \mathcal{C}(\mathcal{L}), \quad L_+ = \mathcal{C}(\mathcal{L}_+).$$

Observe that when  $m$  is a lattice of the form  $\mathcal{P}_{I, -t+1}$ , then  $L = P_{I, t}$ . Both  $L$  and  $L_+$  are normal open compact subgroups of the Iwahori subgroup  $B$ . The quotient  $L/L_+$  is abelian. Further, the characters  $\{L/L_+\}^\wedge$  can be realized as the cosets  $m_+/m$  via formula (1.6). In particular, the coset  $\alpha$  defines a character  $\Omega_\alpha$  of  $L$  trivial on  $L_+$ . The **nondegenerate representations of ramified type** are defined to be the collection of  $(L, \Omega_\alpha)$ 's and their  $G$ -conjugates.

**2. Nondegenerate representations as minimal  $K$ -types.** In this section we show that any admissible irreducible representation  $(\pi, V)$  of  $G$  contains a nondegenerate representation. The approach taken here is analogous to the one used for  $U(2, 1)$  and  $GSp(4)$  in [M1, M2] (see also [KM]). At present it is unclear how to generalize the method to arbitrary  $p$ -adic groups.

Given an irreducible representation  $(\pi, V)$ , define the **level** of  $\pi$  to be the minimum  $i$  such that  $V^{K_i}$ , the space of vectors in  $V$  fixed by the  $i$ th principal congruence subgroup  $K_i$ , is nonzero.

For convenience, assume  $\pi$  has level  $i + 1$ . Choose a nonzero vector  $v \in V^{K_{i+1}}$ , so that  $K_i v$ , the  $K_i$  span of  $v$ , is irreducible. Consider whether  $i = 0$  or  $i \geq 0$ .

*Case  $i = 0$ .* Let  $\sigma$  denote the representation of  $K$  on  $Kv$ . The philosophy of cusp forms [HC] tells us there is a parahoric subgroup  $P \subseteq K$ , and a cuspidal representation  $\Omega$  of  $P/P_1$  so that  $\Omega$  is contained in  $\sigma$ . In particular,  $(P, \Omega)$  is a nondegenerate representation.

*Case  $i \geq 1$ .* Identify, as in §1, the characters of  $K_i/K_{i+1}$  with elements of  $\mathfrak{g}(\mathbb{F}_q)$ . Suppose  $K_i$  acts on  $K_iv$  by the character  $\Omega_\alpha$ . Let  $\alpha = \mathbf{s} + \mathbf{n}$  be the Jordan decomposition of  $\alpha$ . If  $\mathbf{s}$  is nonzero, the pair  $(L, \Omega_\alpha)$ , constructed in §1, is by definition a nondegenerate representation of unramified type and of course contained in  $\pi$ . We are thus reduced to the case when  $\alpha = \mathbf{n}$  is a nilpotent element. If  $\mathbf{n}$  is zero then  $\pi$  has level  $\leq i$  contrary to our hypothesis on the level. We can thus assume  $\mathbf{n}$  is as in (1.8).

We outline our proof. In the following analysis, we define two compact open subgroups  $L_+$  and  $L$ . The two groups will change during the course of the analysis, but the subgroup  $L_+$  will always be normal in  $L$ , with both  $L$  and  $L_+$  normalized by  $B$ . The quotient  $L/L_+$  will always be abelian except possibly when  $i = 1$ . In those cases in which the quotient is abelian, the characters shall be realized, via (1.6), as the cosets  $m_+/m$

$$(2.1) \quad \mathcal{L} = \mathcal{C}^{-1}(L), \quad \mathcal{L}_+ = \mathcal{C}^{-1}(L_+), \quad m = \mathcal{L}^* \text{ and } m_+ = \mathcal{L}_+^*,$$

where

$$\mathcal{C}^{-1}(g) = (g - 1)(1 + g)^{-1}$$

is the inverse Cayley transform. The crux of the argument is to produce a nonzero vector transforming by a nondegenerate representation, or failing that, to produce a  $K_i$  fixed vector, a contradiction.

*Case (1.8a).* The group  $L_+ = B_{6i}$  is contained in the kernel of  $\Omega_n$ . Let  $L = B_{6i-1}$ . In particular, the characters of  $L/L_+$  are realized as the cosets  $\ell^{-6i+1}/\ell^{-6i+2}$ . Such a coset has the form

$$\varpi^{-i}(aE_X + bE_M + \varpi cE_j) \bmod \ell^{-6i+2}, \quad a, b, c \in R.$$

The space  $Lv$  will, of course, decompose into character spaces of  $L$ . The characters which appear in  $Lv$  have  $a = b = 1 \bmod \varpi$ . As already explained, we can assume  $L$  acts on  $v$  by such a character  $\alpha$ . If  $c \in R^\times$ , then  $\alpha$  is nondegenerate of type (1.10a). On the other hand, if  $c \in \varpi$ , then  $v$  is fixed by the group  $L_+ = B_{6i}U_{J,i-1} = P_{\{s_j\}, 5i}$ . In this situation, take  $L = P_{\{s_j\}, 5i-1}$ . A coset  $\mathcal{P}_{\{s_j\}, -5i+1}/\mathcal{P}_{\{s_j\}, -5i+2}$ , i.e., a character in  $\{L/L_+\}^\wedge$ , can be written as

$$\beta = \varpi^{-i}(aE_X + bE_M + \varpi cE_n) \bmod \mathcal{P}_{\{s_j\}, -5i+2}, \quad a, b, c \in R.$$

The parahoric subgroup  $P_{\{s_j\}}$  acts on the character cosets by conjugation. In particular, we have

(2.2)

$$s_j \beta s_j^{-1} = \varpi^{-i}(aE_X + cE_M + -b\varpi E_n) \bmod \mathcal{P}_{\{s_j\}, -5i+2}, \text{ and}$$

$$u_j(\varpi d) \beta u_j(\varpi d)^{-1} = \varpi^{-i}(aE_X + bE_M + \varpi(c + bd)E_n) \bmod \mathcal{P}_{\{s_j\}, -5i+2}, \quad d \in R.$$

Those characters of  $L$  occurring in  $Lv$  have  $a = b = 1 \bmod \varpi$ . Assume  $L$  acts on  $v$  by such a character  $\alpha$ . By (2.2), we can conjugate  $\alpha$  to obtain a character  $\alpha^\#$  whose “ $E_M$ ” coefficient lies in  $\varpi$ . This is done by conjugating by an element in  $U_{j,1}$  to make the coefficient of “ $E_n$ ” belong to  $\varpi$  and then conjugating by  $s_j$ . Let



$w \neq 0$  transform by  $\alpha^\#$ . The characters of  $K_i/K_{i+1}$  appearing in  $K_i w$  have the form  $\Omega_n$ ,  $n$  nilpotent not of type (1.8a).

Cases (1.8b,c,d). Here, the kernel of  $\Omega_n$  contains  $L_+ = P_{\{s_X\}, 3i}$ . Set  $L = P_{\{s_X\}, 3i-1}$ . We have  $m_+ = \rho^{-3i+1}$  and  $m = \rho^{-3i+2}$  (cf. (1.10b,c,d.1)). Write a typical coset  $\beta \in m_+/m$  as

$$\varpi^{-i}(aE_M + bE_V + cE_Z + dE_N + \varpi eE_j) \bmod m.$$

The characters  $\alpha$  of  $L$  in  $Lv$  have

$$(2.3) \quad aE_M + bE_V + cE_Z + dE_N = n \bmod \rho.$$

Fix such a character  $\alpha$ , and a nonzero  $v$  transforming by  $\alpha$ . Consider whether or not  $e$  is a unit. If  $e \in R^\times$ , then  $\alpha$  is nondegenerate of type (1.10b,c,d.1). Consider now  $e \in \rho$ . Here,  $v$  is fixed by the group  $L_+ = P_{\{s_j\}, 2i}$ . Take  $L = P_{\{s_j\}, 2i-1}$ . Write  $\beta \in \varphi^{-2i+1}/\varphi^{-2i+2} = \{L/L_+\}^\wedge$  as

$$(2.4) \quad \beta = \varpi^{-i}\{aE_M + bE_V + cE_Z + dE_N + \varpi(eE_m + fE_y + gE_z + hE_n)\} \bmod \varphi^{-2i+2}.$$

Conjugating  $\beta$  by  $u_j(\varpi u)$ ,  $u \in R$ , we find

$$(2.5) \quad u_j(\varpi u)\beta u_j(\varpi u)^{-1} = \varpi^{-i}\{aE_M + bE_V + cE_Z + dE_N + \varpi(e'E_m + f'E_y + g'E_z + h'E_n)\} \bmod \varphi^{-2i+2},$$

$$e' = e - ud, \quad f' = f - uc, \quad g' = g + ub \quad \text{and} \quad h' = h + ua.$$

Also,  $\beta$  conjugated by  $s_j$  is

$$(2.6) \quad \varpi^{-i}\{hE_M + gE_V + fE_Z + eE_N + \varpi(dE_m + cE_y + bE_z + aE_n)\} \bmod \varphi^{-2i+2}.$$

Let  $v$  be a nonzero vector of  $V$  on which  $L$  acts by an  $\alpha$  satisfying (2.3). Since  $a = 1 \bmod \rho$ , we can further assume, by (2.5), that  $h \in \rho$ . If  $g$  is a unit, then  $\alpha$  is nondegenerate of type (1.10b,c,d.2). If  $g \in \rho$ , then, by (2.6), the characters  $\Omega_n \in \{K_i/K_{i+1}\}^\wedge$  appearing in  $K_i s_j v$  have  $n$  zero or type (1.8e,f).

Case (1.8e). The group  $L_+ = P_{\{s_M\}, 4i}$  is contained in the kernel of  $\Omega_n$ . Let  $L = P_{\{s_M\}, 4i-1}$ . Write a character coset  $\beta \in \mathcal{P}_{\{s_M\}, -4i+1}/\mathcal{P}_{\{s_M\}, -4i+2}$ , as

$$\beta = \varpi^{-i}\{aE_X + bE_V + \varpi(cE_j + dE_n)\} \bmod \mathcal{P}_{\{s_M\}, -4i+2}.$$

For  $t \in R$ , we have

$$u_M(t)\beta u_M(t)^{-1} = \varpi^{-i}\{aE_X + bE_V + \varpi(cE_j + (d - tc)E_n)\} \bmod \mathcal{P}_{\{s_M\}, -4i+2}.$$

The characters in  $Lv$  have  $a, b \in \rho$ . Fix such a character. We claim that we can assume  $c \in \rho$ . Suppose  $c \in R^\times$ . After a conjugation by  $u_M(-d/c)$ , we have  $d \in \rho$ . An additional conjugation by  $s_M$  gives us a character  $\alpha^\#$  whose " $E_j$ ", i.e., " $c$ ", component is in  $\rho$ . The important point is that  $c \in \rho$  implies  $v$  is fixed by the group  $L_+ = P_{\{s_j\}, 2i}$ . Take  $L = P_{\{s_j\}, 2i-1}$ . We now argue in a manner analogous to case (1.8b,c,d). The characters of  $L/L_+$  have the form (2.4). Those characters in  $Lv$  have  $a, b, d \in \rho$  and  $c = 1 \bmod \rho$ . Fix one such character  $\alpha$ . After a conjugation by an element in  $U_{j,1}$  we can assume  $f = 0$ . Consider the  $2 \times 3$  matrix

$$T = \begin{bmatrix} g & 0 & e \\ h & -g & 0 \end{bmatrix}.$$

If  $T \bmod \mathcal{P}$  has rank 2, then  $\alpha$  is nondegenerate of type (1.10e). On the other hand, if  $T$  has rank 1 or 0, it is either

$$\begin{bmatrix} 0 & 0 & 0 \\ h & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & e \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the characters of  $K_i/K_{i+1}$  in  $K_i s_j v$  have the form  $\Omega_n$ ,  $n$  zero or type (1.8f).

*Case (1.8f).* Here, the kernel of  $\Omega_n$  contains  $L_+ = \mathcal{C}(c_{4i-1})$ . Set  $L = \mathcal{C}(c_{4i-2}) = L_+ U_{j,i} U_{j,i-1}$ . The cosets  $c_{-4i+2}/c_{-4i+3} = \{L/L_+\}^\sim$  have the form

$$\varpi^{-i}(aE_J + \varpi bE_j) \bmod c_{-4i+3}, \quad a, b \in R.$$

Let  $L$  act on  $v \neq 0$  by such a character  $\alpha$ . Note that  $a = 1$ . If  $b \neq 0 \bmod \mathcal{P}$ , then  $\alpha$  is nondegenerate of type (1.10f.1). If  $b \in \mathcal{P}$ ,  $v$  is fixed by  $L_+ = P_{\{s_X\}, 3i-1}$ . Set  $L = P_{\{s_X\}, 3i-2}$ . The form of a character coset  $\beta$  here is

$$\varpi^{-i}\{aE_J + \varpi(bE_m + cE_y + dE_z + eE_n)\} \bmod \mathcal{P}^{-3i+3}, \quad a, b, c, d, e \in R.$$

We make two remarks. The parahoric subgroup  $P_{\{s_X\}}$  acts on the  $\beta$ 's. From the discussion after Theorem 3.2 in [C], we see that it is possible to conjugate an arbitrary coset  $\beta$  by  $P_{\{s_X\}}$  into a coset

$$(2.7) \quad \varpi^{-i}(a'E_J + \varpi\theta(n)) \bmod \mathcal{P}^{-3i+3},$$

where

$$a' \in \mathcal{P} \text{ if and only if } a \in \mathcal{P},$$

and  $n \bmod \mathcal{P}$  has the form

$$(2.8) \quad (1.8b, c, d), \quad E_Y, \quad E_M \quad \text{or} \quad 0.$$

Applying the above remarks to  $\alpha$ , we can assume the existence of a nonzero  $v$  which transforms under  $L$  by a coset (2.7) with  $a' \in R^\times$ . We consider separately the four subcases in (2.8).

*Subcase  $n \bmod \mathcal{P}$  of type (1.8b, c, d).* The representation of  $L$  on  $Lv$  is nondegenerate of type (1.10f, b, c, d) respectively.

*Subcase  $n = E_Y \bmod \mathcal{P}$ .* The group  $L_+ = P_{\{s_M\}, 4i-2}$  fixes  $v$ . Let  $L = P_{\{s_M\}, 4i-3}$ . The character cosets  $\{L/L_+\}^\sim$  are

$$\varpi^{-i}\{aE_N + bE_J + \varpi(cE_y + dE_x)\} \bmod \mathcal{P}_{\{s_M\}, -4i+4}, \quad a, b, c, d \in R.$$

Select a  $v$  on which  $L$  acts as a character  $\alpha$  with  $c = -1 \bmod \mathcal{P}$ . After a conjugation, first by an element of  $U_{M,0}$ , and then  $s_M$ , we can assume  $\alpha$  is a character whose " $E_y$ " component is in  $\mathcal{P}$ . Then  $v$  is fixed by  $L_+ = P_{\{s_j\}, 5i-3}$ . Let  $L = P_{\{s_j\}, 5i-4}$ . The character cosets are

$$\varpi^{-i}\{aE_N + \varpi(bE_x + cE_y)\} \bmod \mathcal{P}_{\{s_j\}, -5i+5}, \quad a, b, c \in R.$$

Let  $L$  act on  $v$  by such a character  $\alpha$ . Conjugating by elements in  $U_{j,1}$  and  $s_j$  if necessary, we can assume  $\alpha$  has its " $E_N$ " component in  $\mathcal{P}$ . Consequently,  $v$  is fixed by  $L_+ = B_{6i-4}$ . Set  $L = B_{6i-5}$ , so that  $\{L/L_+\}^\sim = \mathcal{P}_{-6i+5}/\mathcal{P}_{-6i+6}$ . Write

a character coset  $\alpha$  as

$$(2.9) \quad \alpha = \varpi^{-i} \{aE_J + \varpi(bE_m + cE_x)\} \bmod \mathcal{L}_{-6i+6}.$$

Let  $v \neq 0$  transform under  $L$  by such a character. If  $a \in \mathcal{L}$ , then  $v$  is a  $K_i$  fixed vector, a contradiction. Hence we can assume  $a$  is a unit. This is the situation of the last two subcases in (2.8).

*Subcase  $n = E_M \bmod \mathcal{L}$ .* The vector  $v$  is fixed by the group  $L_+ = B_{6i-5}$ . Let  $L = B_{6i-4}$ . Write a character coset  $\alpha$  as in (2.9). Those  $\alpha$ 's in  $Lv$  have  $a = a'$  and  $b = 1 \bmod \mathcal{L}$ . Fix one such  $\alpha$ . If  $c \in R^\times$ , then  $\alpha$  is nondegenerate of type (1.10f.2). If  $c \in \mathcal{L}$ , then  $v$  is fixed by  $L_+ = P_{\{s_M, s_j\}, 3i-2}$ . If  $i = 1$ , then, as in the case of a  $K_1$  fixed vector, we conclude from the philosophy of cusp forms that  $\pi$  contains a level one cuspidal representation of some parabolic subgroup  $P' \subseteq P_{\{s_M, s_j\}}$ . If  $i \geq 2$ , consider the action of  $L = P_{\{s_M, s_j\}, 3i-3}$  on  $v$ . Write a character coset of

$$\{L/L_+\}^\wedge = \mathcal{P}_{\{s_M, s_j\}, -3i+3} / \mathcal{P}_{\{s_M, s_j\}, -3i+2}$$

as

$$\varpi^{-i} \{aE_J + bE_N + \varpi(cE_A + dE_B + eE_M + fE_n) + \varpi^2(gE_n + hE_j)\} \bmod \mathcal{P}_{\{s_M, s_j\}, -3i+2}.$$

We have  $a = a'$ ,  $b = 0$ ,  $e = 1 \bmod \mathcal{L}$  for any character of  $L$  in  $Lv$ . Fix one such character  $\alpha$ . If the characteristic polynomial  $F(t)$  of

$$\begin{bmatrix} c & 0 & -a \\ -g & c-d & e \\ h & f & -d \end{bmatrix} \bmod \mathcal{L}$$

is reducible, then  $\alpha$  is  $P_{\{s_M, s_j\}}$  conjugate to an  $\alpha^\#$ , whose "a" and "b" components belong to  $\mathcal{L}$ . This means  $\pi$  has a nonzero  $K_i$  fixed vector, a contradiction. Hence,  $F(t)$  must be irreducible, i.e.,  $\alpha$  is nondegenerate of type (1.10f.3).

*Subcase  $n = 0 \bmod \mathcal{L}$ .* Here,  $v$  is fixed by  $L_+ = P_{\{s_X, s_j\}, 2i-1}$ . If  $i = 1$ , then by the philosophy of cusp forms once again,  $\pi$  contains a level one cuspidal representation of some parahoric subgroup  $P' \subseteq P_{\{s_X, s_j\}}$ . If  $i \geq 2$ , set  $L = P_{\{s_X, s_j\}, 2i-2}$ . A character coset of

$$\{L/L_+\}^\wedge = \mathcal{P}_{\{s_X, s_j\}, -2i+2} / \mathcal{P}_{\{s_X, s_j\}, -2i+3}$$

has the form

$$\varpi^{-i} \{aE_J + \varpi(gE_A + hE_B + vE_X + wE_x) + \varpi^2 bE_j\} \bmod \mathcal{P}_{\{s_X, s_j\}, -2i+3}.$$

The characters in  $Lv$  satisfy  $a = 1 \bmod \mathcal{L}$ . Assume  $v$  transforms by such a character  $\alpha$ . If  $\delta = (g+h)^2 - 4ab$  is a square  $\bmod \mathcal{L}$ , then

$$u_j(\varpi r)\alpha u_j(\varpi r)^{-1}, \quad r \text{ satisfying } ar^2 + (g+h)r + b = 0 \bmod \mathcal{L},$$

has its "b" coefficient in  $\mathcal{L}$ . Hence,  $H = L_+ U_{J, i-2}$  fixes a nonzero vector. Since  $d(\varpi, \varpi)Hd(\varpi, \varpi)^{-1}$  (cf. (1.2)) contains  $K_i$ , we have a contradiction to the level being  $i+1$ . Thus,  $\delta \bmod \mathcal{L}$  is a nonsquare, i.e.,  $\alpha$  is nondegenerate of type (1.10f.4).

In conclusion, we have shown

**THEOREM 2.1.** *Any irreducible admissible representation of  $G$  contains a nondegenerate representation.*

We turn now to the question of when two nondegenerate representations can occur in the same irreducible representation  $(\pi, V)$  of  $G$ . Define two nondegenerate

representations  $(L, \Omega)$ ,  $(L', \Omega')$  to be **associate** if

- (i)  $L = P$ ,  $L' = P'$  are parahoric subgroups,  $P/P_1 \cong P'/P'_1$  and  $\Omega \cong \Omega'$ , or
- (ii)  $\Omega = \Omega_{\mathbf{s}}$ ,  $\Omega' = \Omega_{\mathbf{s}'}$  with some element of  $\mathbf{s}$  conjugate to some element of  $\mathbf{s}'$ .

**THEOREM 2.2.** *Suppose  $(\pi, V)$  is an irreducible admissible representation of  $G$ . If  $(L, \Omega)$ ,  $(L', \Omega')$  are two nondegenerate representations contained in  $\pi$ , then they are associate.*

**PROOF.** We use the intertwining principle. Let  $W_{\Omega}$  and  $W_{\Omega'}$  respectively be the  $\Omega$  and  $\Omega'$  subspaces of  $V$ . Let  $E_{\Omega'}$  denote the projection from  $V$  onto  $W_{\Omega'}$ . Since  $\pi$  is irreducible, there is a  $g \in G$  such that

$$I = E_{\Omega'} \pi(g): W_{\Omega} \rightarrow W_{\Omega'}$$

is nonzero, and for  $h \in L \cap gL'g^{-1}$ ,  $I\Omega(h) = \Omega'(g^{-1}hg)I$ . We consider three cases according to whether none, one, or both of  $L$ ,  $L'$  are parahoric subgroups.

*Case 1.*  $\Omega = \Omega_{\mathbf{s}}$ ,  $\Omega' = \Omega_{\mathbf{s}'}$ . Write  $\mathbf{s} = s + \mathfrak{m}$ ,  $\mathbf{s}' = s' + \mathfrak{m}'$ . Observe that

$$L \cap gL'g^{-1} = \mathcal{C}(\mathfrak{m}^* \cap g\mathfrak{m}'^*g^{-1}).$$

Therefore, if  $y \in \mathfrak{m}^* \cap g\mathfrak{m}'^*g^{-1}$ , then

$$\psi(\langle y, -s \rangle / 2) = \Omega_{\mathbf{s}}(\mathcal{C}(y)) = \Omega_{\mathbf{s}'}(g^{-1}\mathcal{C}(y)g) = \psi(\langle y, -gs'g^{-1} \rangle / 2).$$

This means  $\text{tr}(y(s - gs'g^{-1})) \in \mathfrak{r}$  for all  $y \in \mathfrak{m}^* \cap g\mathfrak{m}'^*g^{-1}$ . We conclude

$$s - gs'g^{-1} \in \{\mathfrak{m}^* \cap g\mathfrak{m}'^*g^{-1}\}^* = \mathfrak{m} + g\mathfrak{m}'g^{-1},$$

i.e.,  $\mathbf{s}$  and  $gs'g^{-1}$  intersect.

*Case 2.*  $\Omega = \Omega_{\mathbf{s}}$  and  $L' = P'$  a parahoric subgroup. We show  $\Omega$  and  $\Omega'$  cannot both occur in  $\pi$  by showing that  $\Omega$  and the trivial representation of  $P'_1$  cannot occur simultaneously in  $\pi$ . The trivial character of  $P'_1/P'_2$  can be realized as  $\Omega_{\mathbf{s}'}$  with  $\mathbf{s}' = \mathfrak{m}$  and  $P'_1 = \mathcal{C}(\mathfrak{m}^*)$ . By the same reasoning as in case 1, some element of  $\mathbf{s}$  must be conjugate to some element of  $\mathbf{s}'$ . This is impossible because the minimum valuation of the eigenvalues of each element in  $\mathbf{s}$  differ from those of  $\mathbf{s}'$  (see [HM2, Theorem 6.1]).

*Case 3.*  $L$  and  $L'$  both parahoric subgroups. The reasoning again is based on the intertwining principle. It has already been done by Harish-Chandra (see [HC]).

The determination of those irreducibles of  $G$  which contain a given  $(L, \Omega)$  is equivalent to the classifying of the irreducible finite-dimensional representations of the Hecke algebra  $\mathcal{H} = \mathcal{H}(G//L, \Omega^t)$ . The description of  $\mathcal{H}$  will probably follow along the lines given in [HM1, M1, M2]. We hope to return to this in the future.

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